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Path integrals in curved space and the worldline formalism

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We describe, how to construct and compute unambiguously path integrals for particles moving in a curved space, and how these path integrals can be used to calculate Feynman graphs and effective actions for various quantum field theories with external gravity in the framework of the worldline formalism. In particular, we review a recent application of this worldline approach and discuss vector and antisymmetric tensor fields coupled to gravity. This requires the construction of a path integral for the $N = 2$ spinning particle, which is used to compute the first three Seeley–DeWitt coefficients for all p-form gauge fields in all dimensions and to derive exact duality relations.

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1 Introduction

The worldline formalism is an approach based on first quantization which allows to obtain certain QFT results (amplitudes, effective actions, etc.) in a rather simple way. It had been introduced by Feynman in [1] as "an alternative to the formulation of second quantization", and by Schwinger in his famous paper on vacuum polarization [2]. Feynman directly used a path integral approach to describe scalar QED using worldlines of scalar particles, while Schwinger used operatorial quantum mechanical methods to study vacuum effects in QED. Also, the worldline formalism has served as a guide for developing the first quantization of strings, from which it can be recovered as point particle limit, hence the occasional name of "string inspired Feynman rules". Many applications of this formalism and references can be found in the review article [3]. More recent applications include the coupling to external gravitational fields [4–7], studies of string dualities [8], as well as numerical simulations to address nonperturbative issues [9].

In this talk we review the use of the worldline approach to quantum field theories coupled to external gravity, and discuss the main technical tool that is used in such an approach: the path integral for a particle moving in a curved space. This is a subject which has had a longwinded history, with some old controversies fully resolved by now. We end with a brief description of the worldline approach to vector and antisymmetric tensor fields coupled to gravity, which exemplifies the effectiveness of such an approach.

2 The case of a scalar field coupled to gravity

The simplest way to introduce the worldline formalism with background gravity is to consider the example of a scalar field ϕ coupled to the metric $g_{\mu\nu}$. The

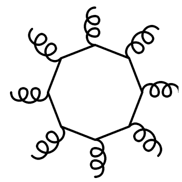


Fig. 1. Loop of a scalar field with external gravitons.

euclidean QFT action reads

$$S[\phi, g] = \int \mathrm{d}^D x \sqrt{g}\, \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2 \right) , \tag{1}$$

where m is the mass of the scalar particle, $g_{\mu\nu}$ is the background metric, and ξ is a nonminimal coupling to the scalar curvature R. The euclidean one-loop effective action $\Gamma[g]$ describes all possible one-loop graphs with the scalar field in the loop and any number of gravitons on the external legs, see Fig. 1.

It can be obtained by path integrating the QFT action $S[\phi, g]$ over ϕ, and is formally given by

$$\mathrm{e}^{-\Gamma[g]} \equiv \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi,g]} = \mathrm{Det}^{-1/2} \left(-\nabla^2 + m^2 + \xi R \right) ,$$

so that

$$\begin{aligned} \Gamma[g] &= -\log \mathrm{Det}^{-1/2} (-\nabla^2 + m^2 + \xi R) = \\ &= \frac{1}{2}\, \mathrm{Tr} \log (-\nabla^2 + m^2 + \xi R) = \\ &= -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T}\, \mathrm{Tr}\, \mathrm{e}^{-T(-\nabla^2 + m^2 + \xi R)} = \\ &= -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}T}{T} \int_{T^1} \mathcal{D}x\, \mathrm{e}^{-S[x;g]} , \end{aligned} \tag{2}$$

where

$$S[x; g] = \int_0^T \mathrm{d}\tau \left(\tfrac{1}{4}\, g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + m^2 + \xi R(x) \right) .$$

In the above equalities $\nabla^2 = g^{\mu\nu} \nabla_\mu \partial_\nu$ is the covariant laplacian acting on scalars. In the third line of (2) we have used the proper time representation of the logarithm,

$$\log \frac{a}{b} = -\int_0^\infty \frac{\mathrm{d}T}{T} \left(\mathrm{e}^{-aT} - \mathrm{e}^{-bT} \right) ,$$

and dropped an additive constant. This provides the starting point of the heat kernel method, originally due to Schwinger and which is by now a well-appreciated tool for studying QFT in curved backgrounds [10,11]. In this method the operator

$\hat{H} = -\nabla^2 + m^2 + \xi R$ is reinterpreted as the quantum hamiltonian of a "fictitious" mechanical model: that of a nonrelativistic particle in curved space with a specific coupling to the scalar curvature. The corresponding Schrödinger equation is then used in trying to solve the problem. However, it proves quite useful to reformulate this quantum mechanics using a path integral: this is shown by the last equality in (2). The exponent of the path integral contains the classical action of the mechanical model whose quantization is expected to produce the quantum hamiltonian $\hat{H}$. The operatorial trace is obtained by using periodic boundary conditions on the worldline time $\tau \in [0, T]$, which therefore describes a one-dimensional torus, or circle, T^1.

It is clear that to use this final path integral formulation one has to be able to define and compute path integrals for particles moving in curved spaces quite precisely. This has been a notoriously complicated and controversial subject. However, this topic is now mature and solid, and will be reviewed in the next section.

For studying other QFT models it is useful to note that the previous effective action for the scalar field can be obtained by first quantizing a scalar point particle with coordinates x^μ and auxiliary einbein e, which is described by the action [12]

$$S[x^\mu, e] = \int_0^1 \mathrm{d}\tau \, \frac{1}{2} \Big[e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e \big(m^2 + \xi R(x) \big) \Big].$$

This worldline action is reparametrization invariant. One can eliminate almost completely the einbein by the gauge condition $e(\tau) = 2T$, and integrate over the remaining modular parameter T after taking into account the correct measure. This reproduces the previous answer in (2). Thus the "fictitious" quantum mechanics mentioned above is not at all that fictitious: it corresponds to the first quantization of the scalar particle which makes the loop in the Feynman graph of Fig. 1. This picture can be enlarged to include particles with spin. Extending the worldline symmetry to $N = 1$ supergravity gives a description of a spin $\frac{1}{2}$ particle, while $N = 2$ supergravity describes particles associated to vector and antisymmetric tensor fields [12–16]. A slightly different approach was discussed in [17]. Gauge fixed versions of these particle models were in fact used to compute gravitational and chiral anomalies in one of the most beautiful applications of the worldline approach [18–20].

Other applications of this worldline approach with external gravity include the computation of trace anomalies [21–24], which in fact was one of the main motivations to study anew path integrals in curved spaces, as well as the calculation of some amplitudes, like the one-loop correction to the graviton propagator due to loops of spin 0, $\frac{1}{2}$, 1 and antisymmetric tensor fields [4, 5, 7] (see Fig. 2).

In [6] the one-loop corrections to the graviton-photon mixing in constant electromagnetic fields due to virtual charged particles has been computed (see Fig. 3). This calculation would have been very difficult to perform using standard methods.

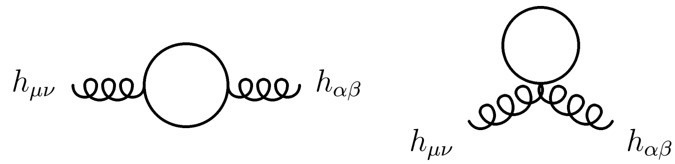


Fig. 2. One-loop matter contributions to the graviton 2-point function.

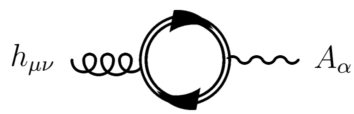


Fig. 3. One-loop correction to graviton-photon mixing in a constant electromagnetic field.

3 Path integrals in curved space: regularizations and counterterms

In this section we wish to discuss, how to construct and compute path integrals for a nonrelativistic particle moving in a curved space. With a slight change in notations (we consider a particle of unit mass which propagates for a total time β, so to have a standard normalization of the action) we consider the following euclidean action

$$S[x] = \int_0^\beta \mathrm{d}t \left(\tfrac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + V(x)\right), \tag{3}$$

where V is an arbitrary scalar potential, from which one wants to construct the path integral

$$Z = \int \mathcal{D}x\, \mathrm{e}^{-S[x]}\,. \tag{4}$$

Construction and computations of path integrals for these nonlinear sigma models can be quite subtle. For example one may find the following description in a well-known textbook on path integrals [25]: "*If you like excitement, conflict and controversy ... then you will love the history of quantization on curved spaces. ... people continue to get signs and factors of 2 wrong in their results.*" That was surely a fair description of the situation at the time that book was written, but now all major difficulties are understood and taken care of, as we are going to describe next.

One dimensional nonlinear sigma models suffer from ordering ambiguities when one applies canonical quantization. The classical hamiltonian reads

$$H = \tfrac{1}{2}\, g^{\mu\nu}(x)p_\mu p_\nu + V(x)$$

and one has to specify an ordering between the p's and the x's. One can impose covariance under change of coordinates at the quantum level, but this selects a subclass of possible orderings and leave unfixed an arbitrary coupling to the scalar

curvature R (the only scalar object that one can construct with two derivative on the metric is the scalar curvature). Thus in the coordinate representation ($p_\mu \to -\mathrm{i}\partial_\mu$) one has a family of covariant quantum hamiltonians

$$\hat{H} = -\tfrac{1}{2}\nabla^2 + \alpha R + V(x)\,,$$

which depend on the parameter α. In the absence of other symmetries that can be used to identify a unique quantum theory, one has to extract the value of α from "experiments", i.e. from the particular physical problem one wishes to describe with the sigma model. For example, in the case discussed in section 2, the scalar relativistic particle, one may demand that conformal invariance holds for vanishing mass, thus fixing $\alpha = \xi/2 = (D-2)/8(D-1)$ in D dimensions. Given this situation, one can always decide to set $\alpha = 0$, and describe additional couplings to R as extra terms contained in the potential V. This is what we will do in the following. The requirement that $\hat{H}$ be covariant and without any coupling to R will be the "renormalization conditions" which will be imposed on the path integral. Equivalently, these conditions can be imposed at the level of the effective action

$$\Gamma = -\log Z = \cdots - \frac{\beta}{12}R + \cdots$$

where Z is the path integral given in (4).

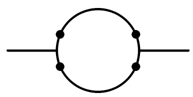
This path integral can be dealt with just as higher dimensional path integrals, i.e. QFT path integrals, where renormalization is needed. Indeed one can always imagine quantum mechanics as a $0+1$ dimensional QFT. Then to compute these path integrals one must use a regularization scheme which consist of: i) a <u>regularization</u>, ii) suitable <u>renormalization conditions</u>, and iii) <u>local counterterms</u> needed to satisfy the renormalization conditions and to eliminate any source of ambiguity. This way any regularization scheme will produce the same correct final answer.

To recognize why a renormalization scheme is necessary, it is enough to notice that the nonlinear sigma model in (3) has derivative interactions which seem to give rise to linear divergences. These divergences can be renormalized away, but that is not necessary. In fact, the covariant path integral measure produces local interactions with additional linear divergences that cancel the previous ones [26]. Thus, one-dimensional nonlinear sigma models are finite. Nevertheless one needs a regularization scheme to handle intermediate divergences and ambiguities. Counterterms are then used to satisfy the renormalization conditions. Since after all the theory is finite, these counterterms are also finite. (One may consider the measure as giving for free the infinite part of the counterterms). Power counting shows that one-dimensional nonlinear sigma models are super-renormalizable, and thus counterterms can only appear up to two-loops.

The previous discussion can be readily exemplified. Let us Taylor expand the metric around the origin $g_{\mu\nu}(x) = g_{\mu\nu}(0) + x^\alpha \partial_\alpha g_{\mu\nu}(0) + \ldots$ and insert this expansion into the action (3). From the leading constant term $g_{\mu\nu}(0)$ one obtains the propagator, which in momentum space and for large momentum k goes like

$$\langle x(k)x(-k)\rangle \sim k^{-2}\,.$$

Then the next term in the expansion of the metric gives a trilinear vertex with two derivatives of the type $x\dot{x}^2$ (we will indicate each derivative by a dot also in Feynman diagrams), so that one can construct the following linearly divergent graph (graphs on the worldline, i.e. in the $0+1$ dimensional QFT)



$$\sim \int \mathrm{d}k \, \frac{k^4}{k^4} \;\to\; \text{linear divergence.}$$

The propagators are in fact compensated by the derivatives that act on each vertex, and this gives rise to a linear divergence.

However, one should consider that the covariant path integral measure carries extra terms. The covariant measure in (4) is formally given by

$$\mathcal{D}x \sim \prod_t \sqrt{g(x(t))}\, \mathrm{d}^D x(t)\,,$$

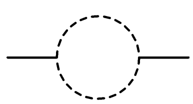
but one can exponentiate the nontrivial $\sqrt{g}$ dependence using commuting a^μ and anticommuting b^μ, c^μ ghost fields with action

$$S_{\text{measure}}[x,a,b,c] = \int_0^\beta \mathrm{d}t \, \tfrac{1}{2}\, g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)\,,$$

so that path integrating over these ghosts reproduces the correct measure. The advantage of this exponentiation is that one can consider perturbatively the effect of the measure, and recognize to which type of diagram they contribute to. One can use the leading term of the Taylor expansion of the metric in the ghost action to identify the ghost propagators, which for large momenta go like

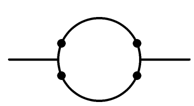
$$\langle a(k)a(-k)\rangle \sim \langle b(k)c(-k)\rangle \sim 1\,.$$

The next term produces a vertex where x couples to the ghosts, and one obtains again a linearly divergent diagram of the type



$$\sim \int \mathrm{d}k \;\to\; \text{linear divergence,}$$

where dashed lines denote ghost propagators. One may check that the previous two diagrams combine to produce a finite result



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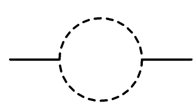


$= \text{finite.}$

The cancellation must of course be achieved carefully: one must regulate each diverging graph and then combine them. Only at this stage one is allowed to

remove the regulator. Different regulators may lead to different left over finite parts. Then different counterterms associated to different regularization schemes make sure that this difference is accounted for to obtain the same correct final answer.

Three different regularization schemes have been developed and checked thoroughly: mode regularization (MR) [21, 27], time slicing (TS) [28], and dimensional regularization (DR) [29]. The precise details of each regularization scheme can be found in the literature, and we will give here only a brief description for each of them.

Mode regularization starts by expanding all fields in Fourier sums. The regularization is achieved by truncating these sums at a fixed mode M, so that all distributions that appear in Feynman graphs become well-behaved functions. Then one performs all computations at finite M, as they are now completely unambiguous (one may check for example that after including the ghosts all possible divergences cancel). Eventually one takes the limit $M \to \infty$, thus obtaining a unique finite result. In practice one can proceed faster: one may perform all manipulations that are valid at the regulated level (for example partial integration) to cast the integrands in alternative forms that can be computed directly in the $M \to \infty$ limit. This scheme requires the addition of a local counterterm $V_{\rm MR}$ to the action (3) to satisfy the renormalization conditions mentioned earlier. This local counterterm is given by

$$V_{\rm MR} = -\frac{1}{8}\,R - \frac{1}{24}\,g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta}\,\Gamma^{\gamma}_{\mu\alpha}\Gamma^{\delta}_{\nu\beta}\,.$$

The noncovariant piece is necessary to restore covariance (which is broken at the regulated level), so that the complete final result is covariant. This regularization is analogous to the standard momentum cut-off used in quantum field theories.

Time slicing is a regularization that is derived from the exact operatorial expression of the transition amplitude. Inserting completeness relations and using the "mid-point prescription" (related to the Weyl ordering of the operators), one derives a discretized path integral in momentum space. By integrating out the momenta and taking the continuum limit, one carefully derives the prescriptions needed for evaluating consistently the products of distributions contained in Feynman diagrams [28]. In particular, the Heaviside step function acquires the value $\theta(0) = \frac{1}{2}$, while Dirac deltas must be used as Kronecker deltas. This regularization requires the counterterm

$$V_{\rm TS} = -\frac{1}{8}\,R + \frac{1}{8}\,g^{\mu\nu}\,\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\,,$$

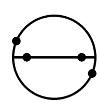
which is seen to arise from Weyl ordering the quantum hamiltonian [30, 31]. Time slicing is a regularization that can be considered analogous to lattice regularization of usual quantum field theories. In [32] it was checked that MR and TS give the same result for the transition amplitude to order β^2, where β is the total propagation time. That calculation produced as byproduct the first three Seeley–DeWitt coefficients for a scalar particle including the corrections for noncoinciding points.

Dimensional regularization is a perturbative regularization which uses an adaptation of standard dimensional regularization to regulate the distributions defined on the compact space $I = [0, \beta]$. One adds d extra infinite dimensions $I \to I \times R^d \equiv \Omega$ and perform all computations of ambiguous Feynman graphs in $d+1$ dimensions. Extra dimensions act as a regulator when d is extended analytically in the complex plane, as usual. After evaluation of the integrals one should take the $d \to 0$ limit. In fact, this is quite difficult since the compact space I produces sums over discrete momenta, and the standard formulas of dimensional regularization do not include such a situation. However there is no need to compute at arbitrary complex d. One may use manipulations valid at the regulated level, like differential equations satisfied by the Green functions and partial integration, to cast the integrand in equivalent forms that, on the other hand, can be unambiguously computed in the $d \to 0$ limit. This method carries a covariant counterterm

$$V_{\rm DR} = -\frac{1}{8}\, R\,. \tag{5}$$

For sigma models with infinite propagation time one can use the standard formulas of dimensional regularization, and in [33] it was originally understood that noncovariant counterterms did not arise. In [34] it was checked that this counterterm is covariant and given by (5).

The previous discussions can be further clarified by going through a specific example. Consider the following superficially logarithmic divergent graph G

$G =$  $= \int_0^1 {\rm d}\tau \int_0^1 {\rm d}\sigma\; {}^\bullet\Delta\;\; {}^\bullet\Delta^\bullet\;\; \Delta^\bullet\,.$

In this example we use Dirichlet boundary conditions $x(0) = x(1) = 0$ for the field $x(\tau)$, where $\tau = t/\beta \in [0,1]$, so that the propagator reads

$$\langle x(\tau) x(\sigma) \rangle = -\beta\, \Delta(\tau, \sigma)$$

with

$$\begin{aligned}
\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty} \left[-\frac{2}{\pi^2 m^2}\, \sin(\pi m \tau) \sin(\pi m \sigma) \right] = \\
&= (\tau - 1)\sigma\, \theta(\tau - \sigma) + (\sigma - 1)\tau\, \theta(\sigma - \tau)\,, \\
{}^\bullet\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty} \left[-\frac{2}{\pi m}\, \cos(\pi m \tau) \sin(\pi m \sigma) \right] = \sigma - \theta(\sigma - \tau)\,, \\
{}^\bullet\Delta^\bullet(\tau,\sigma) &= \sum_{m=1}^{\infty} \Big[-2 \cos(\pi m \tau) \cos(\pi m \sigma) \Big] = 1 - \delta(\tau - \sigma)\,,
\end{aligned}$$

where dots on the left/right indicate derivatives with respect to the first/second variable.

• In mode regularization one cuts off the mode expansion at a big mode number M, and proceeds as follows

$$\begin{aligned}
G(\mathrm{MR}) &= \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ {}^\bullet\!\Delta\ \ {}^\bullet\!\Delta^\bullet\ \ \Delta^\bullet = \frac{1}{2}\int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ \partial_\sigma({}^\bullet\!\Delta^2)\,\Delta^\bullet = \\
&= -\frac{1}{2}\int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ {}^\bullet\!\Delta^2\,(\Delta^{\bullet\bullet}) = -\frac{1}{2}\int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ {}^\bullet\!\Delta^2\,({}^{\bullet\bullet}\!\Delta) = \\
&= -\frac{1}{2}\int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ \frac{1}{3}\,\partial_\tau({}^\bullet\!\Delta^3) = -\frac{1}{6}\int_0^1 \mathrm{d}\sigma\,[{}^\bullet\!\Delta^3(1,\sigma) - {}^\bullet\!\Delta^3(0,\sigma)] \to \\
&\to -\frac{1}{6}\int_0^1 \mathrm{d}\sigma\,[(\sigma-1)^3 - \sigma^3] = -\frac{1}{12}\,.
\end{aligned}$$

• In time slicing one can use $\theta(0) = \frac{1}{2}$, and thus

$$G(\mathrm{TS}) = \int_0^1 \mathrm{d}\tau \int_0^1 \mathrm{d}\sigma\ \Big(\sigma - \theta(\sigma-\tau)\Big)\Big(1 - \delta(\tau-\sigma)\Big)\Big(\tau - \theta(\tau-\sigma)\Big) = -\frac{1}{6}\,.$$

• In dimensional regularization one extends the action to higher dimensions as

$$S = \int_0^1 \mathrm{d}\tau\, \tfrac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \ldots \Rightarrow \int_\Omega \mathrm{d}^{d+1}t\, \tfrac{1}{2}\, g_{\mu\nu}(x)\partial_a x^\mu \partial_a x^\nu + \ldots\,,$$

where the repeated index a is summed from 1 to $d+1$. From this extended action one obtains vertices and propagators, and thus

$$\begin{aligned}
G(\mathrm{DR}) &= \int \mathrm{d}^{D+1}t \int \mathrm{d}^{D+1}s\,({}_a\Delta)\ ({}_a\Delta_b)\ (\Delta_b) = \\
&= \int \mathrm{d}^{D+1}t \int \mathrm{d}^{D+1}s\,({}_a\Delta)\ \partial_a\left(\tfrac{1}{2}\,(\Delta_b)^2\right) = \\
&= -\frac{1}{2}\int \mathrm{d}^{D+1}t \int \mathrm{d}^{d+1}s\,({}_{aa}\Delta)\ (\Delta_b)^2 = \\
&= -\frac{1}{2}\int \mathrm{d}^{D+1}t \int \mathrm{d}^{D+1}s\,\delta^{D+1}(t,s)\ (\Delta_b)^2 = \\
&= -\frac{1}{2}\int \mathrm{d}^{D+1}t\,(\Delta_b)^2|_t \quad\to\quad -\frac{1}{2}\int_0^1 \mathrm{d}\tau\ (\Delta^\bullet)^2\Big|_\tau = -\frac{1}{24}\,,
\end{aligned}$$

where the vertical bar indicates evaluation at coinciding points, and ${}_a\Delta \equiv \partial_a\Delta$. In this computation we used the Green equation ${}_{aa}\Delta(t,s) = \delta^{d+1}(t,s)$ satisfied by the propagator in $d+1$ dimensions.

We have seen concretely, how different regularizations produce different answers. However the different counterterms make sure that the final complete result is independent of the regularization chosen. The conclusion is that path integrals in curved spaces can be defined and computed without any ambiguity.

Let us conclude with another quote, now from a recent book of DeWitt [35], which comments on the extra R terms that appear in the action (the terms that

we now call counterterms): "*Many years ago the author was guilty of suggesting that this term is $\frac{1}{6}R$, a suggestion that remained in the literature for a long time. That the term must be $\frac{1}{8}R$ is conclusively demonstrated in reference [36] where the path integral derivation of the Chern–Gauss–Bonnet formula for the Euler–Poincarè characteristic demands for its consistency.*" This statement witnesses the long lasting confusion on how to calculate in a correct way path integrals in curved spaces. At the same time this statement is rather misleading, as it does not specify how the path integral is computed, i.e. which regularization scheme is used. Most likely DeWitt had in mind a kind of covariant regularization similar to DR (the difference in sign is due to different conventions adopted in the definition of the curvature scalar).

Extensive descriptions, tests and applications of the previous regularization schemes can be found in a forthcoming book [37].

4 $N = 2$ spinning particles and antisymmetric tensor fields

We now describe a recent application of path integrals in curved spaces: the worldline approach to vector and antisymmetric tensor fields coupled to gravity [7]. The worldline action that describes these models is given by the $N = 2$ spinning particle with quantized Chern–Simons coupling [13–16]. This particle is described by phase space coordinates $X = (x^\mu, p_\mu, \psi^\mu, \bar{\psi}_\mu)$ and gauge fields $G = (e, \chi, \bar{\chi}, a)$. The variables ψ^μ, $\bar{\psi}_\mu$, χ, $\bar{\chi}$ are Grassmann variables. The worldline action in flat D dimensional target space is given by

$$S[X,G] = \int \mathrm{d}t \Big(p_\mu \dot{x}^\mu + \mathrm{i}\bar{\psi}_\mu \dot{\psi}^\mu - eH - \mathrm{i}\bar{\chi} Q - \mathrm{i}\chi \bar{Q} - a(J-q) \Big),$$

where the $N = 2$ supersymmetry generators

$$H = \tfrac{1}{2} p_\mu p^\mu\,, \quad Q = p_\mu \psi^\mu\,, \quad \bar{Q} = p_\mu \bar{\psi}^\mu\,, \quad J = \bar{\psi}^\mu \psi_\mu$$

satisfy a first class Poisson-bracket algebra

$$\{Q, \bar{Q}\}_{\mathrm{PB}} = -2\mathrm{i}H\,, \quad \{J, Q\}_{\mathrm{PB}} = \mathrm{i}Q\,, \quad \{J, \bar{Q}\}_{\mathrm{PB}} = -\mathrm{i}\bar{Q}$$

and are gauged by the Lagrange multipliers G. The Chern–Simons coupling q is quantized as $q = \frac{1}{2}D - p - 1$, with p an integer. This model describes a p-form gauge field A_p with field strength $F_{p+1} = \mathrm{d}A_p$ and standard Maxwell action

$$S_p^{\mathrm{QFT}} = \int \mathrm{d}^D x \, \frac{1}{2(p+1)!} \, F_{\mu_1 \cdots \mu_{p+1}} F^{\mu_1 \cdots \mu_{p+1}} \,. \qquad (6)$$

This is immediately seen in canonical quantization, which is introduced by interpreting the phase space coordinates as operators with (anti)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = \mathrm{i}\delta^\mu_\nu\,, \quad \{\hat{\psi}^\mu, \hat{\psi}^\dagger_\nu\} = \delta^\mu_\nu\,.$$

These operators act on wave functions ϕ which depend on the classical configuration space coordinates x^μ and ψ^μ, and thus have an expansion of the form

$$\phi(x,\psi) = F(x) + F_\mu(x)\psi^\mu + \frac{1}{2}\,F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + \ldots + \frac{1}{D!}\,F_{\mu_1\ldots\mu_D}(x)\psi^{\mu_1}\cdots\psi^{\mu_D}\,.$$

The first class constraints now become differential operators

$$\begin{aligned}
\hat H &= -\frac{1}{2}\,\partial_\mu\partial^\mu\,, &\qquad \hat Q &= -\mathrm{i}\psi^\mu\partial_\mu\,,\\
\hat Q^\dagger &= -\mathrm{i}\partial_\mu\,\frac{\partial}{\partial\psi_\mu}\,, &\qquad \hat J &= -\frac{1}{2}\Big[\psi^\mu,\frac{\partial}{\partial\psi^\mu}\Big],
\end{aligned}$$

which select the physical sector of the Hilbert space

$$\begin{aligned}
(\hat J - q)\phi_{\rm phys} = 0 \ &\Rightarrow\ \phi_{\rm phys} \sim F_{\mu_1\ldots\mu_{p+1}}(x)\psi^{\mu_1}\ldots\psi^{\mu_{p+1}}\,,\\
\hat Q\phi_{\rm phys} = 0 \ &\Rightarrow\ \mathrm{d}F_{p+1} = 0\,,\\
\hat{\bar Q}\phi_{\rm phys} = 0 \ &\Rightarrow\ \mathrm{d}^\dagger F_{p+1} = 0\,.
\end{aligned}$$

Thus ones sees that only the tensor $F_{\mu_1\ldots\mu_{p+1}}$ with $p+1$ indices is physical, and must satisfy the Bianchi identities and Maxwell equations. Thus one concludes that the physical sector of the $N=2$ spinning particle describes the field strength of a p-form gauge field with the standard Maxwell action (6).

To introduce the coupling to gravity, we couple the spinning particle to a target space metric $g_{\mu\nu}$ (and corresponding vielbein e_μ^a) preserving the $N=2$ local supersymmetry, then go to configuration space by eliminating p_μ, Wick rotate to euclidean time ($t\to-\mathrm{i}\tau$, and also $a\to\mathrm{i}a$), and obtain the euclidean action

$$\begin{aligned}
S[X,G;g_{\mu\nu}] = \int_0^1 \mathrm{d}\tau\Big[&\frac{1}{2}\,e^{-1}g_{\mu\nu}\left(\dot x^\mu - \bar\chi\psi^\mu - \chi\bar\psi^\mu\right)\left(\dot x^\nu - \bar\chi\psi^\nu - \chi\bar\psi^\nu\right) +\\
&+\bar\psi_a\left(\dot\psi^a + \dot x^\mu\omega_\mu{}^a{}_b\psi^b + \mathrm{i}a\psi^a\right) - \frac{e}{2}\,R_{abcd}\bar\psi^a\psi^b bar\psi^c\psi^d - \mathrm{i}qa\Big].
\end{aligned}$$

The gauge symmetries on the gauge multiplet G are given by

$$\begin{aligned}
\delta e &= \dot\xi + 2\bar\chi\epsilon + 2\chi\bar\epsilon\,,\\
\delta\chi &= \dot\epsilon + \mathrm{i}a\epsilon - \mathrm{i}\alpha\chi\,,\\
\delta\bar\chi &= \dot{\bar\epsilon} - \mathrm{i}a\bar\epsilon + \mathrm{i}\alpha\bar\chi\,,\\
\delta a &= \dot\alpha
\end{aligned}\tag{7}$$

and do not couple to the target space geometry.

The one-loop effective action for a p-form gauge potential has then the following worldline representation

$$\Gamma_p^{\rm QFT}[g_{\mu\nu}] \sim \int_{T^1}\frac{\mathcal{D}G\,\mathcal{D}X}{\mathrm{Vol(Gauge)}}\,\mathrm{e}^{-S[X,G;g_{\mu\nu}]},$$

but first one should fix the gauge symmetries (7). On the one-dimensional torus T^1 we adopt antiperiodic boundary conditions for all fermionic fields. We choose the gauge $\hat{G} = (\beta, 0, 0, \phi)$, insert the Faddeev–Popov determinants, and integrate over the remaining moduli β and ϕ. Fixing appropriately the overall normalization gives

$$\Gamma_p^{\rm QFT}[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{{\rm d}\beta}{\beta}\int_0^{2\pi}\frac{{\rm d}\phi}{2\pi}\left(2\cos\frac{\phi}{2}\right)^{-2}\int_{T^1}\mathcal{D}X\,{\rm e}^{-S[X,\hat{G};g_{\mu\nu}]}, \qquad (8)$$

which contains a path integral of the $N = 2$ nonlinear sigma model

$$Z(\beta, \phi) = \int_{T^1}\mathcal{D}X\,{\rm e}^{-S[X,\hat{G};g_{\mu\nu}]}. \qquad (9)$$

As explained in the previous section, these path integrals are completely under control, and thus one can proceed with concrete applications. An interesting feature of this worldline approach to vector ($p = 1$) and general antisymmetric tensor fields is that on top of the proper time β there is a new modular parameter ϕ. It is related to the Wilson loop variable by

$$w = \exp\left({\rm i}\int_0^1 a{\rm d}\tau\right) = {\rm e}^{{\rm i}\phi}\,.$$

The integration $\phi \in [0, 2\pi]$ has the effect of projecting onto the correct physical sector described by the $(p+1)$-form F_{p+1}. It is interesting to note that the parameter ϕ can be eliminated from the action by a field redefinition of the fermions, which then acquire different boundary conditions: $\psi^a(1) = -{\rm e}^{{\rm i}\phi}\psi^a(0)$. Averaging over ϕ can then be interpreted as averaging over all possible boundary conditions of the fermions. Note that at $\phi = \pi$ a zero mode of the free fermionic kinetic term appears: at this point the fermions have periodic boundary conditions and constant fields ψ_0^a are zero modes.

The worldline representation of the effective action in (8) is quite explicit, and can be used to compute Γ_p^{QFT} in some approximation (the exact evaluation with an arbitrary background metric is impossible to achieve with current techniques). For example, the perturbative evaluation of the path integral for the $N = 2$ nonlinear sigma model in (9) at order β^2 can be carried out without any ambiguity, as already explained in section 2. It allows to identify the first three Seeley–DeWitt coefficients a_0, a_1, a_2 for an arbitrary p-form in arbitrary dimensions. They appear as follows

$$\Gamma_p^{\rm QFT}[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{{\rm d}\beta}{\beta}\int\frac{{\rm d}^Dx\sqrt{g(x)}}{(2\pi\beta)^{D/}}\big(a_0(x) + a_1(x)\beta + a_2(x)\beta^2 + \ldots\big)$$

and their values have been reported in [7]. The cases for $p = 0, 1, 2$ were already known in the literature [10,11], but the cases for $p \geq 3$ are new. These coefficients can in principle be obtained by specializing the known Seeley–DeWitt coefficients for a scalar field coupled to an arbitrary connection to the case under study, and performing the necessary index contractions. However, the latter task is quite

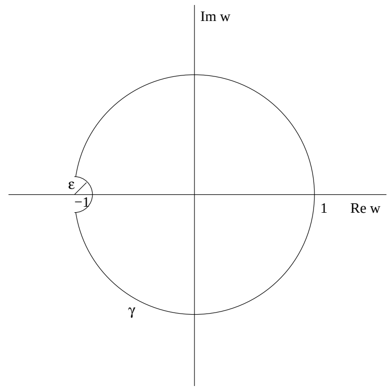


Fig. 4. Regulated contour on the U(1) moduli space.

laborious. The worldline representation maps this problem into the problem of computing some worldline fermion correlators, and makes the computation quite easy and efficient.

A technical point worth of commenting upon is related to the appearance of a singularity on the U(1) moduli space. This singularity appears precisely at the point $\phi = \pi$, where perturbative zero modes arise for the fermions. It is convenient for the present discussion to use the Wilson loop variable w in place of ϕ, and switch to an operatorial picture. Then the effective action in (8) can be rewritten as follows

$$\Gamma_p^{\rm QFT}[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{{\rm d}\beta}{\beta}\oint_\gamma \frac{{\rm d}w}{2\pi {\rm i} w}\,\frac{w}{(1+w)^2}\;{\rm Tr}\left[w^{\hat N-(p+1)}{\rm e}^{-\beta hat H}\right], \qquad (10)$$

where $\hat N$ is the (anti)fermion number operator $\hat\psi^a\hat\psi_a^\dagger$, and the integration region of the Wilson loop variable w is the unit circle γ in the complex w-plane. The singular point $\phi = \pi$ is now mapped to $w = -1$. In particular, the presence of the susy ghost determinant $\frac{w}{(1+w)^2}$ makes this pole rather dangerous. The prescription devised in [7] is to deform the contour to exclude the point $w = -1$, and use contour integration to evaluate the integrals, see Fig. 4.

This prescription permits the calculation of the Seeley–DeWitt coefficients a_0, a_1 and a_2 for all p-forms in arbitrary dimensions, reproducing in particular the known results for the cases of $p = 0, 1, 2$. Moreover, it sheds an interesting light on the issue of quantum equivalence of dual p-forms [38, 39]. A massless p-form is expected to be equivalent to a massless $(D-p-2)$-form. Replacing p with $(D-p-2)$ in (10) gives directly the effective action for a massless $(D-p-2)$-form. This replacement only produces a change $q \to -q$. This change that can be undone by a subsequent change of integration variables $w \to w' = 1/w$. This would seem to prove the exact equivalence. However, the new integration variable has a modified contour of integration which includes the pole, see Fig. 5, so that

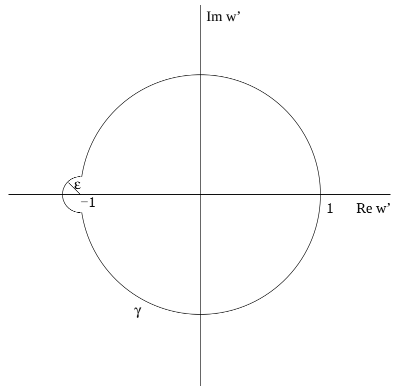


Fig. 5. Contour for the dual differential form.

the total mismatch between the effective actions of dual differential forms is related to the residue at the pole $w = -1$.

In even dimension the mismatch is a local term, proportional to the Euler topological density, which affects that particular Seeley–DeWitt coefficient with the same dimensions of the Euler term. This mismatch was first noticed in four dimensions in [38]. It contributes to the local terms that are usually subtracted when one renormalizes the effective action, and thus, according to [39], it does not really destroy duality. In odd dimension the mismatch is also topological and corresponds to the so-called Ray–Singer torsion, as discovered in [40]. This mismatch can be interpreted as the additional contribution of a $(D-1)$-form gauge potential, which however carries no degrees of freedom. Exact formulas can be found in [7].

In this section we have described an application of the worldline approach to arbitrary antisymmetric tensor fields coupled to gravity. This approach can of course be used to compute some one-loop amplitudes with a certain efficiency as well, see [7]. The particular case $p = 1$ describes a photon coupled to gravity. Previous worldline descriptions of spin 1 particles in $D = 4$ dimensions have been considered in [41] and [42]. In those references only a rigid $N = 2$ linear sigma model was used, together with a limiting procedure necessary to achieve the propagation of the correct degrees of freedom. This limiting procedure is not particularly elegant, but it allows the inclusion of Yang–Mills backgrounds. It is not clear how to include the latter using the $N = 2$ spinning particle with local supersymmetry described above.

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